Worcester Polytechnic Institute August, 2015

Linear Algebra General Comprehensive Exam

Print Name: ______________________ Sign: ______________________

In the following, $\mathbb{R}^n$ is real n-dimensional space, $\mathbb{C}^n$ is complex n-dimensional space, and $\mathbb{R}^{n\times n}$ is the space of real $n \times n$ matrices. If no specific basis is mentioned, then all linear transformations are expressed through the canonical basis.

1. Let $e_1, e_2, ..., e_{2n}$ be the standard basis vectors for $\mathbb{R}^{2n}$. Consider the set S of vectors of the form

$$e_i - e_j,$$

where i is an even index in the range $2, ..., 2n$ and j is an odd index in the range $1, ..., 2n-1$.

a. Is S linearly independent?

b. Is S spanning?

c. What is the orthogonal complement R of the space spanned by S? Compute the dimension and find a basis for R.

2. The ODE $x'' + dx' + kx = 0$ is a simple model of a damped mechanical oscillator with unit mass, in which x is the displacement from equilibrium, $d > 0$ is the drag coefficient, and $k > 0$ is the "spring constant." By introducing a new variable $v = x'$, we can write the ODE as an equivalent system of first-order equations

$$\begin{aligned} x' &= v \\ v' &= -kx - dv \end{aligned}$$

a. Write the system in matrix-vector form $y' = Ay$, where $y \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$.

In the remainder of this problem, take $d = 5$ and $k = 4$.

b. Find the eigenvalues and corresponding eigenvectors of A.

c. Find a nonsingular matrix T and diagonal matrix Λ such that $A = T\Lambda T^{-1}$.

d. If we define $w = T^{-1}y$, what first-order ODE system does w satisfy? What are the advantages of considering this system?

3. Consider a sequence $\{a_n\}_{n=1}^{\infty}$, where a_0 and a_1 are given and subsequent a_n are defined by the recurrence

$$a_n = b\,a_{n-1} + c\,a_{n-2}, \quad n = 2, 3, \ldots$$

for given b and c.

a. Write this recurrence in matrix-vector form

$$v_n = M\,v_{n-1}, \quad n = 2, 3, \ldots,$$

where $v_n = (a_n, a_{n-1})^T$.

In the remainder of this problem, take $b = 3/2$ and $c = 1$.

b. Find the eigenvalues and corresponding eigenvectors of M.

c. Find a pair of values a_0 and a_1 for which $a_n \to \infty$, and another pair for which $a_n \to 0$.

4. Suppose that $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

a. What are the eigenvalues of A and of A^{-1}?

b. What are the eigenvalues of $(A^3 + 3A^2 + 3A + I)^{-1}$?

5. a. For $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$, find a least-squares solution of $Ax = b$. Is your solution unique?

b. What is the dimension of $\mathcal{S} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix} \right\}$?

c. Find an orthonormal basis of $\mathcal{S}$.

d. Determine the orthogonal projection of $v = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ onto $\mathcal{S}$.

6. For $A \in \mathbb{R}^{n \times n}$, define $\langle u, v \rangle = u^T A v$ for all u and v in $\mathbb{R}^n$.

a. What is a necessary and sufficient condition on A for $\langle \cdot, \cdot \rangle$ to be an inner product on $\mathbb{R}^n$?

For the remainder of this problem, assume that this condition holds and that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$.

b. What is a necessary and sufficient condition on $M \in \mathbb{R}^{n \times n}$ for multiplication by M to be a self-adjoint transformation on $\mathbb{R}^n$ with respect to $\langle \cdot, \cdot \rangle$?

c. Suppose that $\mathcal{S}$ is a k-dimensional subspace of $\mathbb{R}^{n \times n}$ and that the columns of $U \in \mathbb{R}^{n \times k}$ are orthonormal with respect to $\langle \cdot, \cdot \rangle$. Show that the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(v) = UU^T Av$ is orthogonal projection onto $\mathcal{S}$ with respect to $\langle \cdot, \cdot \rangle$.

7. Let A be a real $n \times n$ matrix with an eigenvalue λ having algebraic multiplicity n. Show that

$$e^{At} = e^{\lambda t}\left[I + (A - \lambda I)t + \ldots + \frac{(A - \lambda I)^{n-1}}{(n-1)!} t^{n-1} \right]$$

(hint: show that both side satisfy the same ODE)